Don Blazys Polygonal Number Problem

Formulation

Starting from OEIS:A090466 "Regular figurative or polygonal numbers of order greater than 2" = (6, 9, 10, 12, 15, 16, 18, 21, 22, 24, 25, 27, 28, 30, 33, 34, 35, 36, 39, 40, 42, 45, 46, 48, 49, 51, 52, 54, 55, 57, 58, 60, 63, 64, 65, 66, 69, 70, 72, 75, 76, 78, 81, 82, 84, 85, 87, 88, 90, 91, 92, 93, 94, 95, 96, 99, 100, 102, 105, 106, 108, 111, 112, 114, 115, 117, 118),
Don defines the function
$\overline{\omega}(x)$ as the count of numbers in A090466 that are <= *x*.

For example, we see that $\overline{\omega}(10) = 3$ and $\overline{\omega}(100) = 57$.

Algorithm

Some numbers in the sequence occur for just one polygonal number, whereas others occur for several polygonal numbers of different ranks.

The outline of an algorithm is to create an array of *x* entries, then generate all polygonal numbers of all ranks that yield numbers <= *x*.

Corresponding entries in the array are marked, and finally the marked entries are counted, giving $\overline{\omega}(x)$.

Obviously, for large *x*, some coding tricks will be required since most computers today cannot allocate arrays of more than a few GByte.

Computed values

Computed values for the first powers of 10:

Power	x = 10^{Power}	Computed w(x)
1	10	3
2	100	57
3	1000	622
4	10000	6357
5	100000	63889
6	1000000	639946
7	10000000	6402325
8	100000000	64032121
9	1000000000	640349979
10	10000000000	6403587409
11	100000000000	64036148166
12	1000000000000	640362343980
13	10000000000000	6403626146905
14	100000000000000	64036270046655
15	1000000000000000	640362727589917

The last value took several weeks of CPU time to arrive at on a modern computer.

Don's approximations

In his paper, Don suggests 2 approximations to $\overline{\varpi}(x)$.

The first is an equation of the form

$\overline{\varpi}(x) = \overline{\varpi} = x - \frac{x}{F} - \frac{1}{2}\sqrt{x - \frac{x}{F}}$, where $F = \alpha\pi e + e$, and $\alpha \approx 137.035999084^{-1}$ is the "fine structure constant".

Let $f = 1 - \frac{1}{F}$, so

$$\boxed{\overline{\varpi}_1(x) = xf - \frac{1}{2}\sqrt{xf}} \qquad (1)$$

The second approximation has a multiplicative correction constant *c=0.00000398606454280 61*.

$$\boxed{\overline{\varpi}_2(x) = c\left(xf - \frac{1}{2}\sqrt{xf}\right)} \qquad (2)$$

Results:

Power	x	w	Calculated w1	Calculated w2	Error 1	Error 2
1	10	3	5	5	2	2
2	100	57	60	60	3	3
3	1000	622	628	628	6	6
4	10000	6357	6364	6364	7	7
5	100000	63889	63910	63910	21	21
6	1000000	639946	639965	639963	19	17
7	10000000	6402325	6402388	6402362	63	37
8	100000000	64032121	64032528	64032273	407	152
9	1000000000	640349979	640352643	640350090	2664	111
10	10000000000	6403587409	6403612945	6403587420	25536	11
11	100000000000	64036148166	64036403036	64036147783	254870	-383
12	1000000000000	640362343980	640364895519	640362342983	2551539	-997
13	10000000000000	6403626146905	6403651691060	6403626165691	25544155	18786
14	100000000000000	64036270046655	64036525562183	64036270308459	255515528	261804
15	1000000000000000	640362727589917	640365282980521	640362730443172	2555390604	2853255

For both approximations the error increases sharply for the larger values of *x*.

Two other approximations

Solving (1) for *f*:

$$\sqrt{xf} = 2\left(xf - \overline{\omega}\right)$$

In the original equation the positive root has been implicitly chosen so we must have $\boxed{xf \geq \overline{\omega}}$.

$$xf = 4\left(xf - \overline{\omega}\right)^2$$

$$xf = 4x^2 f^2 - 8xf\overline{\omega} + 4\overline{\omega}^2$$

$$4x^2 f^2 - xf(1 + 8\overline{\omega}) + 4\overline{\omega}^2 = 0$$

Solving this second degree equation gives:

$$f = \frac{x(1+8\overline{\omega}) \pm \sqrt{x^2(1+8\overline{\omega})^2 - 64x^2\overline{\omega}^2}}{8x^2}$$

A factor *x* can be removed:

$$f = \frac{(1+8\overline{\omega}) \pm \sqrt{(1+8\overline{\omega})^2 - 64\overline{\omega}^2}}{8x}$$

Simplifying under the square root:

$$\boxed{f = \frac{(1+8\overline{\omega}) \pm \sqrt{1+16\overline{\omega}}}{8x}}$$

Which sign should we select for the square root?

We know from above that $xf \geq \overline{\varpi}$ so we must have

$$xf = \frac{(1+8\overline{\varpi}) \pm \sqrt{1+16\overline{\varpi}}}{8x} x \geq \overline{\varpi}$$

$$\frac{(1+8\overline{\varpi}) \pm \sqrt{1+16\overline{\varpi}}}{8} \geq \overline{\varpi}$$

$$(\frac{1}{8}+\overline{\varpi}) \pm \sqrt{\frac{1}{64}+\frac{1}{4}\overline{\varpi}} \geq \overline{\varpi}$$

$$\frac{1}{8} \pm \sqrt{\frac{1}{64}+\frac{1}{4}\overline{\varpi}} \geq 0$$

Since $\sqrt{\frac{1}{64}} = \frac{1}{8}$, we can write $\frac{1}{8} \pm \left(\frac{1}{8}+p\right) \geq 0$, where *p>0* for $\overline{\varpi} > 0$

Choosing the negative sign leads to the contradiction $-p \geq 0$. The positive sign is therefore the correct choice.

So we get:

$$\boxed{f = \frac{(1+8\overline{\varpi}) + \sqrt{1+16\overline{\varpi}}}{8x}} \qquad \text{(a)}$$

When used where $x, \overline{\varpi} >> 1$, further simplification yields:

$$\boxed{f = \frac{8\overline{\varpi} + 4\sqrt{\overline{\varpi}}}{8x} = \frac{2\overline{\varpi} + \sqrt{\overline{\varpi}}}{2x} \quad x, \overline{\varpi} >> 1}$$

And using $F = \frac{1}{1-f}$ gives the value of *F* and *1/alfa*.

A solution in closed form to Don's second approximation has not been found.

Numerical results

Using the formula (a) above we obtain the following values

Power	x	w	f	alfa	1/alfa
1	10	3	0.40000000000	-0.12314378110	-8.12058872194
2	100	57	0.60901986232	-0.01880704252	-53.17157116875
3	1000	622	0.63459559040	0.00215604136	463.81299560847
4	10000	6357	0.63969905945	0.00669526887	149.35919971818
5	100000	63889	0.64015506429	0.00710712383	140.70389418721
6	1000000	639946	0.64034610814	0.00727998146	137.36298709483
7	10000000	6402325	0.64035902658	0.00729167677	137.14266705813
8	100000000	64032121	0.64036122129	0.00729366377	137.10530566060
9	1000000000	640349979	0.64036263169	0.00729494070	137.08130620943
10	10000000000	6403587409	0.64036274203	0.00729504060	137.07942901827
11	100000000000	64036148166	0.64036274693	0.00729504503	137.07934572331
12	1000000000000	640362343980	0.64036274409	0.00729504246	137.07939397345
13	10000000000000	6403626146905	0.64036274122	0.00729503986	137.07944289989
14	100000000000000	64036270046655	0.64036274048	0.00729503919	137.07945548180
15	1000000000000000	640362727589917	0.64036274024	0.00729503898	137.07945948440

The smallest mean square error (*25650*) occurs for *1/alfa=137.0794594430*, corresponding to a value $f=f_0=0.640362740245$.

Power	x	w(x)	Calculated w3	Error
1	10	3	5	2
2	100	57	60	3
3	1000	622	628	6
4	10000	6357	6364	7
5	100000	63889	63910	21
6	1000000	639946	639963	17
7	10000000	6402325	6402362	37
8	100000000	64032121	64032273	152
9	1000000000	640349979	640350088	109
10	10000000000	6403587409	6403587391	-18
11	100000000000	64036148166	64036147498	-668
12	1000000000000	640362343980	640362340132	-3848
13	10000000000000	6403626146905	6403626137181	-9724
14	100000000000000	64036270046655	64036270023371	-23284
15	1000000000000000	640362727589917	640362727592350	2433

The errors are small considering that *w* becomes very large.

A good approximation is therefore

$$\boxed{\overline{\omega}_3(x) = xf_0 - \frac{1}{2}\sqrt{xf_0} \quad \mathrm{f}_0 = 0.640362740245} \qquad (3)$$

<u>The "Fine structure constant" calculated from this value of f_0 is *137.0794594430* which differs from the physically determined value *137.035999084* by around *0.04*, or *0.03%.*</u>

For increasing *x*, the *sqrt* term becomes less important so an even simpler approximation is:

$$\overline{\omega}_4(x) = xf_1 \quad f_1 = 0.640362727310 \qquad (4)$$

with a mean square error of *2952390*.

Power	x	w(x)	Calculated w4	Error
1	10	3	6	3
2	100	57	64	7
3	1000	622	640	18
4	10000	6357	6404	47
5	100000	63889	64036	147
6	1000000	639946	640363	417
7	10000000	6402325	6403627	1302
8	100000000	64032121	64036273	4152
9	1000000000	640349979	640362727	12748
10	10000000000	6403587409	6403627273	39864
11	100000000000	64036148166	64036272731	124565
12	1000000000000	640362343980	640362727310	383330
13	10000000000000	6403626146905	6403627273100	1126195
14	100000000000000	64036270046655	64036272731000	2684345
15	1000000000000000	640362727589917	640362727310000	-279917

The error values for w_4 have the same order of magnitude as those of w_2.

Form of the expression

As can be seen from the table of errors of formula (3), the errors tend to go increasingly negative except for the last one (*Power=15*) which has a small positive value.

The same happens when *Power=15* is left out of the calculations, now *Power=14* has a small positive value while *Power=10..13* show increasing negative values.

This suggests that the right hand expression of (3) has not the correct form, but that we need to look for another expression in order to get a better approximation.

Conclusion

The values of $\overline{\omega}(x)$ calculated so far can fairly well approximated by:

$$\overline{\omega}_3(x) = xf_0 - \frac{1}{2}\sqrt{xf_0} \quad f_0 = 0.640362740245$$

Reasoning purely from the data available, f_0 cannot be associated with "Fine structure constant".

Of course it is possible to create expressions involving f_0 together with e, pi, proton to electron mass ratio, or any number of other dimensionless constants to get a result slightly above *137*.

But that would be just a play with numbers which does not prove anything.